10B.3 Heat conduction in a nuclear fuel rod assembly

The differential equation may be set up following the procedure described in the lecture notes, or Section 10.2 of BSL. Then, when Fourier's law with constant thermal conductivity is substituted into the heat balance equation, we get

\[-k_F \frac{d}{dr} \left( r \frac{dT_F}{dr} \right) = S_{n0} \left[ 1 + b \left( \frac{r}{R_F} \right)^2 \right] r\]

for the heat conduction equation in the fuel rod. In the cladding a similar equation, without the source term, is appropriate:

\[-k_C \frac{d}{dr} \left( r \frac{dT_C}{dr} \right) = 0\]

The boundary conditions in this problem are

B. C. 1: At \( r = 0 \), \( T_F \) is finite
B. C. 2: At \( r = R_F \), \( T_F = T_C \)
B. C. 3: At \( r = R_F \), \(-k_F \left( dT_F / dr \right) = -k_C \left( dT_C / dr \right)\)
B. C. 4: At \( r = R_C \), \(-k_C \left( dT_C / dr \right) = h_L \left( T_C - T_L \right)\)

Integrating the above differential equations twice gives

\[\frac{dT_F}{dr} = -\frac{S_{n0} r}{2k_F} \left( 1 + \frac{b}{R_F^2} \frac{r^2}{2} \right) + \frac{C_1}{r}\]
\[\frac{dT_C}{dr} = \frac{C_3}{r}\]

\[T_F = -\frac{S_{n0} r^2}{4k_F} \left( 1 + \frac{b}{R_F^2} \frac{r^2}{4} \right) + C_1 \ln r + C_2; \quad T_C = C_3 \ln r + C_4\]

The constant is zero by B. C. 1, since the temperature is not infinite at the axis of the fuel rod. From B. C. 3, we can find \( C_3 \):

\[C_3 = -\frac{S_{n0} R_F^2}{2k_C} \left( 1 + \frac{b}{2} \right)\]
From B. C. 4, we get $C_4$:

$$C_4 = T_L - \left( \frac{k_C}{R_C h_L} + \ln R_C \right) C_3 = T_L + \left( \frac{k_C}{R_C h_L} + \ln R_C \right) \frac{S_{n0} R_F^2}{2 k_C} \left( 1 + \frac{b}{2} \right)$$

And finally $C_2$ can be obtained from B. C. 2:

$$C_2 = T_L + \frac{S_{n0} R_F^2}{4 k_F} \left( 1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2 k_C} \left( 1 + \frac{b}{2} \right) \left( \ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$

Then we can get the maximum temperature at the axis of the fuel rod:

$$T_{F,\text{max}} = T_L + \frac{S_{n0} R_F^2}{4 k_F} \left( 1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2 k_C} \left( 1 + \frac{b}{2} \right) \left( \ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$
10B.4 Heat conduction in an annulus

a. The energy balance on a cylindrical shell of thickness $\Delta r$ and length $L$ is

$$2\pi r L q_r \bigg|_r - 2\pi (r + \Delta r) L q_r \bigg|_{r+\Delta r} = 0 \quad \text{or} \quad 2\pi L (r q_r) \bigg|_r - 2\pi L (r q_r) \bigg|_{r+\Delta r} = 0$$

When this equation is divided by $2\pi L$ and the limit is taken as $\Delta r$ goes to zero, we get

$$\frac{d}{dr} (rq_r) = 0$$

which may be integrated to give

$$rq_r = C_1 \quad \text{or} \quad \frac{k}{r} \frac{dT}{dr} = \frac{C}{r}$$

The thermal conductivity varies linearly with temperature, so that

$$k = k_0 + (k_1 - k_0) \frac{T - T_0}{T_1 - T_0} = k_0 + (k_1 - k_0) \Theta$$

Then

$$-\left[ k_0 + (k_1 - k_0) \Theta \right] \frac{dT}{dr} = \frac{C_1}{r} \quad \text{or} \quad -(T_1 - T_0) [k_0 + (k_1 - k_0) \Theta] \frac{d\Theta}{dr} = \frac{C_1}{r}$$

This first-order, separable differential equation may be integrated:

$$-(T_1 - T_0) [k_0 + \frac{1}{2} (k_1 - k_0) \Theta] \Theta = C_1 \ln r + C_2$$

The constants of integration may be found from the boundary conditions: $\Theta(r_0) = 0$ and $\Theta(r_1) = 1$.

$$0 = C_1 \ln r_0 + C_2 \quad \text{and} \quad -(T_1 - T_0) [k_0 + \frac{1}{2} (k_1 - k_0)] = C_1 \ln r_1 + C_2$$

When these relations are subtracted, and equation for $C_1$ is obtained:
\[
C_1 = -\frac{(T_1 - T_0)}{\ln(r_1/r_0)} \left[ \frac{1}{2} (k_0 + k_1) \right]
\]

and \(C_2\) may also be obtained if desired.

The heat flow through the wall may then be obtained:

\[
Q = 2\pi r_0 L q_r \bigg|_{r=r_0} = 2\pi L r_0 \left( \frac{C_1}{r_0} \right) = 2\pi L \frac{(T_0 - T_1)}{\ln(r_1/r_0)} \left[ \frac{1}{2} (k_0 + k_1) \right]
\]

b. Let the ratio of the outer to the inner radius be written as \(r_1/r_0 = 1 + \varepsilon\), where \(\varepsilon\) is very small. Then use the Taylor series for the logarithm as given in Eq. C.2-3: \(\ln(1 + \varepsilon) = \varepsilon - \frac{1}{2} \varepsilon^2 + \frac{1}{3} \varepsilon^3 - \ldots\). If we keep just one term of the series, then this corresponds to

\[
\varepsilon = (r_1/r_0) - 1 = (r_1 - r_0)/r_0
\]

When this is substituted into the expression for \(Q\) we get

\[
Q = 2\pi L r_0 \left[ \frac{1}{2} (k_0 + k_1) \right] \frac{T_0 - T_1}{r_1 - r_0}
\]

This is just: area times average thermal conductivity times a temperature gradient.
10B.8 Electrical heating of a pipe

For the assumptions made in the problem statement we may assume that temperature is a function only of \( r \) and that

\[
q_r |_{r=R} = -k \frac{\partial T}{\partial r} |_{r=R} = 0
\]

therefore that all generated heat must leave from the surface at \( R \) by Newton’s law of cooling:

\[
V S_e = Ah(T_1 - T_s)
\]
or

\[
T(r = R) = T_1 = \frac{T_a + R(1 - \kappa^2)S_e}{2h}
\]

These relations provide a basis for setting up a differential equation as well as giving the boundary conditions for integrating it. The next step is a shell balance over the region between any radial position “\( r \)” and a nearby position “\( r + \Delta r \)”:

\[
2\pi r L \frac{q_r |_{r}}{\Delta r} + 2\pi r L S_e \Delta r = 2\pi r L q_r |_{r+\Delta r}
\]
or

\[
\frac{1}{r} \left[ r q_r |_{r+\Delta r} - r q_r |_{r} \right] \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} r q_r = S_e
\]

It follows, just as for Ex. 10.2, that

\[
q_r = -k \frac{\partial T}{\partial r} = \frac{r S_e}{2} + \frac{C_1}{r}
\]

However, the above boundary condition now gives

\[
C_1 = -\frac{(\kappa R)^2 S_e}{2}
\]

and

\[
\frac{\partial T}{\partial r} = -\frac{r S_e}{2k} + \frac{1}{r} \frac{(\kappa R)^2 S_e}{2k}
\]

\[
T = -\frac{r^2 S_e}{4k} + \ln r \cdot \frac{(\kappa R)^2 S_e}{2k} + C_2
\]

Therefore

\[
T = T_1 + \frac{R^2 S_e}{4k} \left\{ \left[ 1 - \left( \frac{r}{R} \right)^2 \right] - 2\kappa^2 \ln \left( \frac{R}{r} \right) \right\}
\]

and \( T_1 \) is related to the surrounding temperature \( T_s \) by the above equation.
T is the temperature profile in the pipe, and the desired temperature of inner pipe surface is \( T_\kappa \). Substitute this condition into the expression of temperature profile:

\[
T(r = \kappa R) = T_\kappa = T_1 + \frac{SeR^2}{4k} [(1 - \kappa^2) + 2\kappa^2 \ln \kappa]
\]

where

\[
T_1 = T_a + \frac{SeR}{2h} (1 - \kappa^2)
\]

\[
\Rightarrow T_\kappa = T_a + \frac{SeR}{2h} (1 - \kappa^2) + \frac{SeR^2}{4k} [(1 - \kappa^2) + 2\kappa^2 \ln \kappa]
\]

\[
\Rightarrow Se = \frac{T_\kappa - T_a}{\frac{(1 - \kappa^2)R^2}{2h} - \frac{(\kappa R)^2}{4k} \left[ (1 - \frac{1}{\kappa^2}) - 2\ln \kappa \right]}
\]

Assume the electrical power is converted to heat without any loss of energy, then the total electrical power \( P \) is:

\[
P = V \cdot Se
\]

\[
= \pi (1 - \kappa^2) R^2 L \cdot \frac{T_\kappa - T_a}{\frac{(1 - \kappa^2)R^2}{2h} - \frac{(\kappa R)^2}{4k} \left[ (1 - \frac{1}{\kappa^2}) - 2\ln \kappa \right]}
\]

\[
= \frac{\pi R^2 (1 - \kappa^2) L(T_\kappa - T_a)}{\frac{(1 - \kappa^2)R^2}{2h} - \frac{(\kappa R)^2}{4k} \left[ (1 - \frac{1}{\kappa^2}) - 2\ln \kappa \right]}
\]