ADV. TRANSPORT II 2012 Problem set 2 solutions:

Corrections due Feb 7, 5 pm together with solutions of set 3.

A1. Water evaporation, it is one dimensional, steady-state diffusion

\[ j = \frac{DH}{l} \Delta c \]

The flux \( j \) can be calculated by the height change, \( \Delta h = 6.22 \frac{cm}{5 \text{ days}} = 1.44 \cdot 10^{-5} \frac{cm}{sec} \)

\[ j = \Delta h \cdot \text{density} \cdot \text{molecular/molecular weight} = 1.44 \cdot 10^{-5} \frac{cm}{sec} \cdot 1 \frac{g}{cm^3} \cdot \frac{1\text{mol}}{18.015 \text{g}} = 8.0 \cdot 10^{-7} \frac{mol}{cm^2 \text{sec}} \]

The concentration difference \( \Delta c \) can be found from pressure. First, we calculate the difference in the partial pressure of water across the air film. Saturated vapour pressure \( p_{sat} = 2338.8 \text{ Pa} \). Since the air above the water is 100% saturated:

\[ \Delta p = p_{sat} - 0.5 p_{sat} = 0.5 \cdot 2338.8 = 1169.4 \text{ Pa} \]

Air can be assumed as ideal gas, and by ideal gas law:

\[ \Delta c = \frac{\Delta p}{RT} = \frac{1169.4 \text{ Pa}}{8.3145 \frac{J}{K \cdot \text{mol}}} \left( \frac{293.15 \text{ K}}{1 \text{ mol}} \right) = 0.48 \frac{\text{mol}}{m^3} = 4.8 \cdot 10^{-7} \frac{\text{mol}}{cm^3} \]

We can now calculate the permeability of this air film:

\[ DH = \frac{j \cdot l}{\Delta c} = \frac{8.0 \cdot 10^{-7} \frac{\text{mol}}{cm^2 \text{sec}} \cdot 0.15 \text{ cm}}{4.8 \cdot 10^{-7} \frac{\text{mol}}{cm^3}} = 0.25 \frac{cm^2}{\text{sec}} \]

A2. Twin-bulb method; it is with the concept as in Diaphragm-cell diffusion.

We define the direction of positive flux is from bulb A into bulb B. A balance on bulb A gives:

\[ V \frac{dc_1}{dt} = -(\pi r^2) j_1 \]

Similarly for bulb B we have:

\[ V \frac{dc_2}{dt} = (\pi r^2) j_1 \]

Subtracting the second equation from the first, we get:
The flux is given by:

\[ j_1 = \frac{DH}{\ell} (c_1 - c_2) \]

In this case \( H = 1 \) since there is no interface. Combining these equations and substituting \( \Delta c \) for \( c_1 - c_2 \) we have:

\[
V \frac{d\Delta c}{dt} = \frac{-2\pi Dr^2}{\Delta c} \Rightarrow \frac{d\Delta c}{\Delta c} = \frac{-2\pi Dr^2}{\ell V} dt
\]

Integration gives:

\[
\int_{\Delta c_0}^{\Delta c} \frac{d\Delta c}{\Delta c} = \int_0^{\infty} \frac{-2\pi Dr^2 t}{\ell V} dt \Rightarrow \Delta c = \Delta c_0 e^{\frac{-2\pi Dr^2 t}{\ell V}}
\]

**A3. Carburizing of steel; unsteady-state diffusion**

Rearranging Eq. (2.3-15) we have:

\[
c_1 = c_{10} + (c_{1\infty} - c_{10}) \text{erf}\left(\frac{z}{\sqrt{4Dt}}\right)
\]

Since this is a semiinfinite slab, \( c_{1\infty} = 0 \). Assuming the argument of the error function is small (say less than about 0.7), we can approximate it as the argument itself:

\[
c_1 \approx c_{10} - \left(\frac{c_{10}}{\sqrt{4Dt}}\right)z
\]

Using the data for 10 hr we calculate the slope from its endpoints:

\[
m = \frac{0.7 - 1.35}{(0.05 - 0) \text{in}} = -5.12 \text{cm}^{-1}
\]

The data show that the concentration \( c_1 \) at 10 hr is given by:

\[
c_1 = 1.35 - 5.12z
\]

By comparison with our approximation we have:
\[ c_{10} = 1.35 \]
\[ \frac{c_{10}}{\sqrt{4Dt}} = 5.12 \]

Solving for \( D \) gives:
\[ D = \frac{\left( \frac{c_{10}}{5.12} \right)^2}{4t} = \frac{\left( \frac{1.35}{5.12 \text{ cm}^{-1}} \right)^2}{4 \left( 10 \text{ hr} \cdot \frac{3600 \text{ s}}{\text{hr}} \right)} = 4.8 \cdot 10^{-7} \text{ cm}^2 / \text{s} \]

To check whether our approximation was valid, we calculate the argument of the error function:
\[ \frac{z}{\sqrt{4Dt}} = \frac{0.05 \text{ in} \cdot \frac{2.54 \text{ cm}}{\text{in}}}{\sqrt{4 \left( 4.8 \cdot 10^{-7} \text{ cm}^2 / \text{s} \right) \left( 10 \text{ hr} \cdot \frac{3600 \text{ s}}{\text{hr}} \right)}} = 0.48 \]

The assumption is valid.

**A4. Diffusion of dopant in arsenide; from the decay of a pulse (see lecture notes or 2.4.1 in Cussler)**

From eq. 2.4-14, we have:
\[ c = \frac{M}{A} \frac{e^{-z^2/4Dt}}{\sqrt{4\pi Dt}} \]

Since the maximum concentration will be at \( z = 0 \) i.e. the site of the scratch we can write:
\[ c_{\text{max}} = \frac{M}{A} \frac{e^{-z^2/4Dt}}{\sqrt{4\pi Dt}} \to \frac{c}{c_{\text{max}}} = \exp \left( -\frac{z^2}{4Dt} \right) = 0.291 \]

Solving for \( t \), we have:
\[ t = \frac{z^2}{4Dt} \ln \left( \frac{c_{\text{max}}}{c_1} \right) = \frac{\left( 4 \cdot 10^{-4} \text{ cm} \right)^2}{4 \left( 9 \cdot 10^{-12} \text{ cm}^2 / \text{s} \right) \ln \left( \frac{1}{0.291} \right)} = 3600 \text{ s} = 1 \text{ hour} \]

**A5. Steady-state flux out of a pipe with porous wall**

Eq. (2.4-29) is the balanced equation for unsteady diffusion into cylinders.
\[ \frac{\partial c_1}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left( r \frac{\partial c_1}{\partial r} \right) \]

For steady-state diffusion, the time derivative is zero.

\[ 0 = \frac{D}{r} \frac{d}{dr} \left( r \frac{dc_1}{dr} \right) \Rightarrow \frac{d}{dr} \left( r \frac{dc_1}{dr} \right) = 0 \]

Integrating twice we have:

\[ c_1 = A \ln r + B \]

With boundary conditions, we can solve constant \( A \) and \( B \)

\[ r = R_i \quad c = c_{i_i} \]
\[ r = R_o \quad c = 0 \]

Applying the second and then the first we have:

\[ 0 = A \ln R_o + B \Rightarrow B = -A \ln R_o \]

\[ c_{i_i} = A \ln R_i + B = A \ln R_i - A \ln R_o = A \ln \frac{R_i}{R_o} \Rightarrow A = \frac{c_{i_i}}{\ln \frac{R_i}{R_o}} \]

\[ B = -A \ln R_o = -\frac{c_{i_i} \ln R_o}{\ln \frac{R_i}{R_o}} \]

Substitution gives:

\[ c_1 = \frac{c_{i_i} \ln r}{\ln \frac{R_i}{R_o}} - \frac{c_{i_i} \ln R_o}{\ln \frac{R_i}{R_o}} \]

\[ = c_{i_i} \frac{\ln r}{\ln \frac{R_i}{R_o}} \]

\[ = c_{i_i} \frac{R_i}{R_o} \]

The flux at the outside of the pipe is given by Fick’s Law:

\[ j = -D \frac{dc_1}{dr} \bigg|_{r=R_o} = -\frac{Dc_{i_i}}{r \ln \frac{R_i}{R_o}} \bigg|_{r=R_o} = \frac{Dc_{i_i}}{R_o \ln \frac{R_o}{R_i}} \]

Steady-state diffusion across a thin slab (Cussler 2.2.1)

Mass balance (2.2-3)
Boundary conditions (2.2-6, 2.2-7)
Concentration (2.2-9)
Flux (2.2-10)

Steady-state diffusion away from a dissolving sphere (Cussler 2.4.2)
Mass balance (2.4-18)
Boundary conditions (2.4-19, 2.4-20)
Concentration (2.4-23)
Flux (2.4-25)